

# THE MONOTONICITY RESULTS AND SHARP INEQUALITIES FOR SOME POWER-TYPE MEANS OF TWO ARGUMENTS

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ABSTRACT. For  $a, b > 0$  with  $a \neq b$ , we define

$$M_p = M^{1/p}(a^p, b^p) \text{ if } p \neq 0 \text{ and } M_0 = \sqrt{ab},$$

where  $M = A, He, L, I, P, T, N, Z$  and  $Y$  stand for the arithmetic mean, Heronian mean, logarithmic mean, identric (exponential) mean, the first Seiffert mean, the second Seiffert mean, Neuman-Sándor mean, power-exponential mean and exponential-geometric mean, respectively. Generally, if  $M$  is a mean of  $a$  and  $b$ , then  $M_p$  is also, and call "power-type mean". We prove the power-type means  $P_p, T_p, N_p, Z_p$  are increasing in  $p$  on  $\mathbb{R}$  and establish sharp inequalities among power-type means  $A_p, He_p, L_p, I_p, P_p, N_p, Z_p, Y_p$ . From this a very nice chain of inequalities for these means

$$L_2 < P < N_{1/2} < He < A_{2/3} < I < Z_{1/3} < Y_{1/2}.$$

follows. Lastly, a conjecture is proposed.

## 1. INTRODUCTION

There are many basic bivariate means of positive numbers  $a$  and  $b$ , such as

- the arithmetic mean  $A$  defined by

$$(1.1) \quad A(a, b) = \frac{a+b}{2};$$

- geometric mean  $G$  defined as

$$(1.2) \quad G(a, b) = \sqrt{ab};$$

- Heronian mean  $He$  defined by

$$(1.3) \quad He(a, b) = \frac{a+b+\sqrt{ab}}{3};$$

- logarithmic mean  $L$  defined by

$$(1.4) \quad L(a, b) = \frac{a-b}{\ln a - \ln b} \text{ if } a \neq b \text{ and } L(a, a) = a;$$

- identric (exponential) mean defined by

$$(1.5) \quad I(a, b) = e^{-1} \left( \frac{a^a}{b^b} \right)^{1/(a-b)} \text{ if } a \neq b \text{ and } I(a, a) = a;$$

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- the first Seiffert mean  $P$ , defined in [17] as

$$(1.6) \quad P(a, b) = \frac{a-b}{2 \arcsin \frac{a-b}{a+b}} \text{ if } a \neq b \text{ and } P(a, a) = a;$$

- the second Seiffert mean  $T$ , defined in [18] by

$$(1.7) \quad T(a, b) = \frac{a-b}{2 \arctan \frac{a-b}{a+b}} \text{ if } a \neq b \text{ and } T(a, a) = a;$$

- Neuman-Sándor mean  $N$ , defined in [11] by

$$(1.8) \quad N(a, b) = \frac{a-b}{2 \operatorname{arcsinh} \frac{a-b}{a+b}} \text{ if } a \neq b \text{ and } N(a, a) = a;$$

- power-exponential mean, the special case of Gini means [5], defined by

$$(1.9) \quad Z(a, b) = a^{\frac{a}{a+b}} b^{\frac{b}{a+b}};$$

- exponential-geometric mean, defined in [19] by

$$(1.10) \quad Y(a, b) = I \exp \left( 1 - \frac{G^2}{L^2} \right),$$

where  $I, G, L$  denote the identric mean, geometric mean and logarithmic mean of positive numbers  $a$  and  $b$ .

We define

$$(1.11) \quad M_p := M_p(a, b) = M^{1/p}(a^p, b^p) \text{ if } p \neq 0 \text{ and } M_0 = \sqrt{ab},$$

where  $M = A, He, L, I, P, T, N, Z$  and  $Y$  stand for the arithmetic mean, Heroian mean, logarithmic mean, identric (exponential) mean, the first Seiffert mean, the second Seiffert mean, Neuman-Sándor mean, power-exponential mean and exponential-geometric mean, which are defined by (1.1)- (1.10), respectively. It is known that  $A_p$  is the classical power mean which is increasing with  $p$  on  $\mathbb{R}$ .

Also, we note that for  $t \in \mathbb{R}$

$$(1.12) \quad M_{pt}^t(a, b) = M^{1/p}(a^{pt}, b^{pt}) = M_p(a^t, b^t).$$

In the most cases, ones more prefer to evaluate a given more complicated mean  $M$  by a simpler one such as arithmetic mean, geometric mean, Heroian mean and power mean etc. For example, for  $a, b > 0$  with  $a \neq b$ , Lin [10] gave a best estimation of the logarithmic mean  $L$  by power means, that is,

$$(1.13) \quad A_0(a, b) < L(a, b) < A_{1/3}(a, b).$$

Jiao and Cau proved in [9] that

$$(1.14) \quad L(a, b) < He_{1/2}(a, b) < A_{1/3}(a, b).$$

Stolarsky [14] and Pittenger [13] showed that the inequalities

$$(1.15) \quad A_{2/3}(a, b) < I(a, b) < A_{\ln 2}(a, b)$$

hold, where the constants  $2/3$  and  $\ln 2$  are the best possible. The following sharp double inequality

$$(1.16) \quad A_{\ln 2 / \ln 3}(a, b) < He(a, b) < A_{2/3}(a, b)$$

is due to Alzer and Janous [2].

For the first Seiffert mean, Jagers first established in [8] (also see [6]) that

$$A_{1/2}(a, b) < P(a, b) < A_{2/3}(a, b),$$

which has been improved by Hästö in [7] as

$$(1.17) \quad A_{\log_{\pi} 2}(a, b) < P(a, b) < A_{2/3}(a, b),$$

where  $\log_{\pi} 2$  and  $2/3$  are the best possible. In 1995, Seiffert [18] indicated that

$$(1.18) \quad A(a, b) < T(a, b) < A_2(a, b),$$

which was refined by Yang [22] as

$$(1.19) \quad A_{\log_{\pi/2} 2}(a, b) < T(a, b) < A_{5/3}(a, b),$$

where  $\log_{\pi/2} 2$  and  $3/5$  can not be improved. Utilizing (1.12) the second one of 1.19 can be written as

$$(1.20) \quad T_{2/5}(a, b) < A_{2/3}(a, b).$$

Chu et al. showed in [3] an optimal double inequality

$$(1.21) \quad He_{\frac{\ln 3}{\ln(\pi/2)}} < T(a, b) < He_{5/2}(a, b),$$

the second one in which is equivalent to

$$(1.22) \quad T_{2/5}(a, b) < He(a, b).$$

For the Neuman-Sándor mean, Yang [23] has presented sharp bounds in terms of power means, that is,

$$(1.23) \quad A_{\frac{\ln 2}{\ln \ln(3+2\sqrt{2})}}(a, b) < N(a, b) < A_{4/3}(a, b),$$

which by using (1.12) implies that

$$(1.24) \quad N_{1/2}(a, b) < A_{2/3}(a, b).$$

For the power-exponential mean  $Z$ , from the comparison theorem for Gini means given by Páles in [12] (also see [1], [16], [20]) it is easy to obtain the following optimal inequality:

$$(1.25) \quad Z(a, b) > A_2(a, b),$$

which also can be written as

$$(1.26) \quad Z_{1/3}(a, b) > A_{2/3}(a, b),$$

On the other hand, Sándor showed in [15] that

$$(1.27) \quad L < P < I.$$

Neuman and Sándor [11] established the following chain of inequalities for means:

$$G < L < P < A < N < T < A_2.$$

The following chain of inequalities for means

$$(1.28) \quad L_2 < He < A_{2/3} < I < Z_{1/3} < Y_{1/2}$$

is due to Yang [20, (5.17)]. Recently, Costin and Toader [4] presented a nice separation of some Seiffert type means by power means:

$$(1.29) \quad G < L < A_{1/2} < P < A < N < T < A_2,$$

which has been improved by Yang [23] as

$$(1.30) \quad \begin{aligned} A_0 &< L < A_{1/3} < A_{\ln_\pi 2} < P < A_{2/3} < I < A_{\ln 2} \\ &< A_{\frac{\ln 2}{\ln \ln(3+2\sqrt{2})}} < N < A_{4/3} < A_{\log_{\pi/2} 2} < T < A_{5/3}. \end{aligned}$$

Motivated by these inequalities for bivariate means, the purpose of this paper is to investigate the monotonicities of  $P_p, T_p, N_p, Z_p$  in  $p$  (in the next section, we will prove that  $M_p$  is also a mean and call it "power-type mean") and establish the relations among  $M_p$  defined by (1.11).

## 2. THE MONOTONICITIES OF POWER-TYPE MEANS

In general, a function  $M : \mathbb{R}_+^2 \mapsto \mathbb{R}$  is called a bivariate mean if

$$\min(a, b) \leq M(a, b) \leq \max(a, b)$$

holds for all  $a, b > 0$ . Clearly, each bivariate mean is reflexive, that is,

$$M(a, a) = a \text{ for all } a > 0.$$

A bivariate mean is symmetric if

$$M(a, b) = M(b, a)$$

holds for all  $a, b > 0$ . It is said to be homogeneous (of degree one) if

$$M(ta, tb) = tM(a, b)$$

holds for all  $a, b, t > 0$ .

Let  $M$  be a differentiable mean on  $\mathbb{R}_+^2$ . Now we introduce the function  $M_p : \mathbb{R}_+^2 \mapsto \mathbb{R}$  defined by

$$(2.1) \quad M_p(a, b) = M^{1/p}(a^p, b^p) \text{ if } p \neq 0 \text{ and } M_0(a, b) = a^{M_x(1,1)} b^{M_y(1,1)},$$

where  $M_x(x, y)$ ,  $M_y(x, y)$  stand for the first-order partial derivatives with respect to the first and second component of  $M(x, y)$ , respectively.

The following theorem reveals that  $M_p$  is also a mean, and it is called "M mean of order  $p$ ". Since the form of  $M_p$  is similar to power mean  $A_p$ , it is also known simply as "power-type mean".

**Theorem 1.** *Let  $M$  be a differentiable mean on  $\mathbb{R}_+^2$  and  $M_p$  be defined by (2.1). Then  $M_p$  is also a mean. In particular,  $M_0 = G$  if  $M$  is symmetric.*

*Proof.* We distinguish two cases to prove it.

Case 1:  $p \neq 0$ . Without loss of generality, we assume that  $p > 0$  and  $b > a > 0$ . Since  $M$  is a mean, we have  $a^p < M(a^p, b^p) < b^p$ , which implies that  $a < M^{1/p}(a^p, b^p) < b$ , that is,  $M^{1/p}(a^p, b^p)$  is also a mean.

Case 2:  $p = 0$ . Clearly, it suffices to show that

$$M_x(1, 1), M_y(1, 1) \in (0, 1) \text{ and } M_x(1, 1) + M_y(1, 1) = 1.$$

In fact, it is known that

$$\begin{aligned} x &< M(x + \Delta x, x) < x + \Delta x \text{ if } \Delta x > 0, \\ x + \Delta x &< M(x + \Delta x, x) < x \text{ if } \Delta x < 0, \end{aligned}$$

which can be written as

$$0 < \frac{M(x + \Delta x, x) - x}{\Delta x} < 1.$$

And so,

$$M_x(x, x) = \lim_{\Delta x \rightarrow 0} \frac{M(x + \Delta x, x) - M(x, x)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{M(x + \Delta x, x) - x}{\Delta x} \in (0, 1).$$

Similarly, we can prove  $M_y(x, x) \in (0, 1)$ .

Differentiating for the identity  $M(x, x) = x$  yields  $M_x(x, x) + M_y(x, x) = 1$ . Hence,  $M_0$  is still a mean.

Particularly, if  $M$  is symmetric, that is,  $M(x, y) = M(y, x)$ , then it is easy to obtain that  $M_x(x, y) = M_y(y, x)$ . From this it is deduced that  $M_x(x, x) = M_y(x, x)$ , which together with  $M_x(x, x) + M_y(x, x) = 1$  leads to  $M_x(x, x) = M_y(x, x) = 1/2$ . Thus,  $M_0 = G$ .

This completes the proof.  $\square$

Applying the results in [19], we can give a sufficient condition for the monotonicity of  $p$ -order  $M$  mean  $M_p$ .

**Lemma 1.** *Let  $M$  be a homogeneous and differentiable mean. Then the function  $M_p$  defined by (2.1) is strictly increasing (decreasing) in  $p$  on  $\mathbb{R}$  if  $\mathcal{I} = (\ln M)_{xy} > (<)0$ .*

For  $M = A, He, L, I, Y$ , it has been proven that  $\mathcal{I} = (\ln M)_{xy} < (>)0$  in [21], and so all the corresponding  $p$ -order arithmetic mean (i.e., power mean)  $A_p$ ,  $p$ -order Heroian mean  $He_p$ ,  $p$ -order logarithmic mean  $L_p$ ,  $p$ -order identric (exponential) mean  $I_p$  and  $p$ -order exponential-geometric mean  $Y_p$  are strictly increasing in  $p$  on  $\mathbb{R}$ . Now we shall show that the first  $p$ -order Seiffert mean  $P_p$ , the second  $p$ -order Seiffert mean  $T_p$ ,  $p$ -order Neuman-Sándor mean  $N_p$  and  $p$ -order power-exponential mean  $Z_p$  have the same monotonicity.

**Theorem 2.** *The first  $p$ -order Seiffert mean  $P_p$ , the second  $p$ -order Seiffert mean  $T_p$ ,  $p$ -order Neuman-Sándor mean  $N_p$  and  $p$ -order power-exponential mean  $Z_p$  are strictly increasing in  $p$  on  $\mathbb{R}$ .*

*Proof.* By Theorem 1, it suffices to show that  $\mathcal{I} = (\ln M)_{xy} < 0$ , where  $M = P, T, N$ .

(i) Direct computation yields

$$\begin{aligned} \mathcal{I} &= (\ln P)_{xy} = \frac{1}{(x-y)^2} - \frac{1}{\arcsin^2 \frac{x-y}{x+y}} \frac{1}{(x+y)^2} - \frac{1}{2} \frac{1}{\arcsin \frac{x-y}{x+y}} \frac{x-y}{\sqrt{xy}(x+y)^2} \\ &= \frac{1}{(x-y)^2} - \frac{4}{(x+y)^2(x-y)^2} P^2 - \frac{1}{(x+y)^2 \sqrt{xy}} P. \end{aligned}$$

Using the known inequality  $P > G = \sqrt{xy}$ , we get

$$\mathcal{I} < \frac{1}{(x-y)^2} - \frac{4}{(x+y)^2(x-y)^2} xy - \frac{1}{(x+y)^2 \sqrt{xy}} \sqrt{xy} = 0.$$

(ii) In the same way, we have

$$\begin{aligned} \mathcal{I} &= (\ln T)_{xy} = \frac{1}{(x-y)^2} - \frac{y}{\arctan^2 \frac{x-y}{x+y}} \frac{x}{(x^2+y^2)^2} - \frac{1}{\arctan \frac{x-y}{x+y}} \frac{x^2-y^2}{(x^2+y^2)^2} \\ &= \frac{1}{(x-y)^2} - \frac{4xy}{(x^2+y^2)^2(x-y)^2} T^2 - \frac{2(x+y)}{(x^2+y^2)^2} T. \end{aligned}$$

The known inequality  $T > A = (x + y)/2$  results in

$$\begin{aligned}\mathcal{I} &< \frac{1}{(x-y)^2} - \frac{4xy}{(x^2+y^2)^2(x-y)^2} \left(\frac{x+y}{2}\right)^2 - \frac{2(x+y)}{(x^2+y^2)^2} \left(\frac{x+y}{2}\right) \\ &= -\frac{xy}{(x^2+y^2)^2} < 0.\end{aligned}$$

(iii) We have

$$\begin{aligned}\mathcal{I} &= (\ln N)_{xy} = \frac{1}{(x-y)^2} - \frac{1}{\operatorname{arcsinh}^2 \frac{x-y}{x+y}} \frac{2xy}{(x^2+y^2)(x+y)^2} - \frac{\sqrt{2}(x^2+y^2+xy)}{(x+y)^2(\sqrt{x^2+y^2})^3} \frac{x-y}{\operatorname{arcsinh} \frac{x-y}{x+y}} \\ &= \frac{1}{(x-y)^2} - \frac{8xy}{(x-y)^2(x^2+y^2)(x+y)^2} N^2 - \frac{2\sqrt{2}(x^2+y^2+xy)}{(x+y)^2(\sqrt{x^2+y^2})^3} N.\end{aligned}$$

Application of the inequality

$$N > \frac{A^2}{A_2} = \frac{\left(\frac{x+y}{2}\right)^2}{\sqrt{\frac{x^2+y^2}{2}}}$$

proved in [23] leads to

$$\mathcal{I} = (\ln N)_{xy} < \frac{1}{(x-y)^2} - \frac{8xy}{(x-y)^2(x^2+y^2)(x+y)^2} \left(\frac{\left(\frac{x+y}{2}\right)^2}{\sqrt{\frac{x^2+y^2}{2}}}\right)^2 - \frac{2\sqrt{2}(x^2+y^2+xy)}{(x+y)^2(\sqrt{x^2+y^2})^3} \frac{\left(\frac{x+y}{2}\right)^2}{\sqrt{\frac{x^2+y^2}{2}}} = 0.$$

(iv) Lastly, we prove the monotonicity of  $Z_p$  in  $p$ . To this end, it suffices to prove that the function

$$p \mapsto \ln Z_p = \frac{a^p}{a^p + b^p} \ln a + \frac{b^p}{a^p + b^p} \ln b$$

is increasing on  $\mathbb{R}$ . In fact, we have

$$\frac{d}{dp} (\ln Z_p) = a^p b^p \frac{(\ln a - \ln b)^2}{(a^p + b^p)^2} > 0,$$

which proves the monotonicity of  $Z_p$  in  $p$  and the whole proof is completed.  $\square$

### 3. SHARP INEQUALITIES AMONG POWER-TYPE MEANS

We first establish the relation between logarithmic mean of order  $p$  and the first Seiffert mean.

**Theorem 3.** *For  $a, b > 0$  with  $a \neq b$ , the inequality  $L_p < P$  if and only if  $p \leq 2$ .*

*Proof.* Due to the symmetry, we assume that  $a < b$ . Then inequality  $L_p(a, b) < P(a, b)$  is equivalent with

$$(3.1) \quad \left(\frac{x^p - 1}{p \ln x}\right)^{1/p} < \frac{x - 1}{2 \arcsin \frac{x-1}{x+1}},$$

where  $x = a/b \in (0, 1)$ .

Necessity. If  $L_p < P$ , then we have

$$\lim_{x \rightarrow 1} \frac{\left(\frac{x^p - 1}{p \ln x}\right)^{1/p} - \frac{x - 1}{2 \arcsin \frac{x-1}{x+1}}}{(x - 1)^2} = \frac{1}{24}p - \frac{1}{12} \leq 0,$$

which indicates that  $p \leq 2$ .

Sufficiency. We prove the inequality (3.1) holds if  $p \leq 2$ . By Theorem 2, it suffices to show that the inequality (3.1) holds if  $p = 2$ . Let the function  $f_1$  be defined on  $(0, 1)$  by

$$f_1(x) = 2 \frac{(x+1)}{x-1} \arcsin^2 \frac{x-1}{x+1} - \ln x.$$

Differentiation yields

$$\begin{aligned} f_1'(x) &= -4 \frac{\arcsin^2 \frac{x-1}{x+1}}{(x-1)^2} + 4 \left( \arcsin \frac{x-1}{x+1} \right) \frac{1}{\sqrt{x}(x-1)} - \frac{1}{x} \\ &= - \frac{\left( x-1 - 2\sqrt{x} \arcsin \frac{x-1}{x+1} \right)^2}{x(x-1)^2} < 0, \end{aligned}$$

which shows that  $f_1$  is decreasing on  $(0, 1)$ . Hence  $f_1(x) > \lim_{x \rightarrow 1^-} f_1(x) = 0$ , and so

$$\frac{x^2 - 1}{2 \ln x} < \frac{(x-1)^2}{4 \left( \arcsin \frac{x-1}{x+1} \right)^2},$$

this proves the sufficiency and the proof is complete.  $\square$

Secondly, we show the relation between the first Seiffert mean of order  $p$  and Neuman-Sándor mean.

**Theorem 4.** For  $a, b > 0$  with  $a \neq b$ , the inequality  $P_p < N$  if and only if  $p \leq 2$ .

*Proof.* Similarly, we assume that  $a < b$ . Then inequality  $P_p(a, b) < N(a, b)$  is equivalent with

$$(3.2) \quad \left( \frac{x^p - 1}{2 \arcsin \frac{x^p - 1}{x^p + 1}} \right)^{1/p} < \frac{x-1}{2 \ln \frac{x-1 + \sqrt{2(x^2+1)}}{x+1}},$$

where  $x = a/b \in (0, 1)$ .

Necessity. If  $P_p < N$ , then we have

$$\lim_{x \rightarrow 1} \frac{\left( \frac{x^p - 1}{2 \arcsin \frac{x^p - 1}{x^p + 1}} \right)^{1/p} - \frac{x-1}{2 \ln \frac{x-1 + \sqrt{2(x^2+1)}}{x+1}}}{(x-1)^2} = \frac{1}{12}p - \frac{1}{6} \leq 0,$$

which implies that  $p \leq 2$ .

Sufficiency. We prove the inequality (3.2) holds if  $p \leq 2$ . By Theorem 2, it suffices to show that the inequality (3.2) holds if  $p = 2$ . We define the function  $f_2$  by

$$f_2(x) = 2 \frac{\ln^2 \frac{x-1 + \sqrt{2(x^2+1)}}{x+1}}{x-1} (x+1) - \arcsin \frac{x^2-1}{x^2+1}, \quad x \in (0, 1).$$

Differentiation yields

$$\begin{aligned} f_2'(x) &= -4 \frac{\ln^2 \frac{x-1+\sqrt{2(x^2+1)}}{x+1}}{(x-1)^2} + 4 \frac{\ln \frac{x-1+\sqrt{2(x^2+1)}}{x+1}}{x-1} \frac{\sqrt{2}}{\sqrt{x^2+1}} - \frac{2}{x^2+1} \\ &= - \left( 2 \frac{\ln \frac{x-1+\sqrt{2(x^2+1)}}{x+1}}{x-1} - \frac{\sqrt{2}}{\sqrt{x^2+1}} \right)^2 < 0, \end{aligned}$$

which implies that  $f_2$  is decreasing on  $(0, 1)$ . Therefore  $f_2(x) > \lim_{x \rightarrow 1^-} f_2(x) = 0$ , and then

$$\frac{x^2-1}{2 \arcsin \frac{x^2-1}{x^2+1}} < \left( \frac{x-1}{2 \ln \frac{x-1+\sqrt{2(x^2+1)}}{x+1}} \right)^2,$$

this proves the sufficiency and the proof is finished.  $\square$

Thirdly, let us prove the inequality for Neuman-Sandor mean and Heronian mean of order  $p$ .

**Theorem 5.** *For  $a, b > 0$  with  $a \neq b$ , the inequality  $N < He_p$  if and only if  $p \geq 2$ .*

*Proof.* We assume that  $a < b$ . Then inequality  $N(a, b) < He_p(a, b)$  is equivalent with

$$(3.3) \quad \frac{x-1}{2 \ln \frac{x-1+\sqrt{2(x^2+1)}}{x+1}} < \left( \frac{x^p + x^{p/2} + 1}{3} \right)^{1/p},$$

where  $x = a/b \in (0, 1)$ .

Necessity. If  $N < He_p$ , then we have

$$\lim_{x \rightarrow 1} \frac{\frac{x-1}{2 \ln \frac{x-1+\sqrt{2(x^2+1)}}{x+1}} - \left( \frac{x^p + x^{p/2} + 1}{3} \right)^{1/p}}{(x-1)^2} = \frac{1}{6} - \frac{1}{12}p \leq 0,$$

which reveals that  $p \geq 2$ .

Sufficiency. We prove the inequality (3.3) holds if  $p \geq 2$ . By Theorem 2, it suffices to show that the inequality (3.3) holds if  $p = 2$ . To this end, we define the function  $f_3$  by

$$f_3(x) = \frac{x-1}{\sqrt{\frac{x^2+x+1}{3}}} - 2 \ln \frac{x-1+\sqrt{2(x^2+1)}}{x+1}.$$



Differentiation yields

$$\begin{aligned}
f'_3(x) &= \frac{1}{\sqrt{\frac{x^2+x+1}{3}}} - \frac{2x+1}{6} \frac{x-1}{\left(\frac{x^2+x+1}{3}\right)^{\frac{3}{2}}} - \frac{2\sqrt{2}}{(x+1)\sqrt{x^2+1}} \\
&= \frac{\sqrt{2}x\sqrt{\frac{x^2+1}{2}} - 2\left(\frac{x^2+x+1}{3}\right)^{3/2} + \left(\frac{x^2+1}{2}\right)^{3/2}}{(x+1)\sqrt{x^2+1}\left(\sqrt{\frac{x^2+x+1}{3}}\right)^3} \\
&= -\sqrt{2} \frac{\left(\sqrt{\frac{x^2+1}{2}} - \sqrt{\frac{1}{3}x + \frac{1}{3}x^2 + \frac{1}{3}}\right)^2 \left(\sqrt{\frac{x^2+1}{2}} + 2\sqrt{\frac{1}{3}x + \frac{1}{3}x^2 + \frac{1}{3}}\right)}{(x+1)\sqrt{x^2+1}\left(\sqrt{\frac{x^2+x+1}{3}}\right)^3} < 0,
\end{aligned}$$

which shows that  $f_3$  is decreasing on  $(0, 1)$ . Hence  $f_3(x) > \lim_{x \rightarrow 1^-} f_3(x) = 0$ , and so

$$\frac{x-1}{\sqrt{\frac{x^2+x+1}{3}}} > 2 \ln \frac{x-1 + \sqrt{2(x^2+1)}}{x+1},$$

which implies that the inequality (3.3) holds if  $p = 2$ , that is, the sufficiency holds.

Thus the proof ends.  $\square$

Next we further prove the sharp inequality for identric (exponential) mean and power-exponential mean of order  $p$ .

**Theorem 6.** For  $a, b > 0$  with  $a \neq b$ , the inequality  $I < Z_p$  if and only if  $p \geq 1/3$ .

*Proof.* We assume that  $a < b$ . Then inequality  $I(a, b) < Z_p(a, b)$  is equivalent with

$$(3.4) \quad e^{-1}x^{x/(x-1)} < x^{\frac{x^p}{x^p+1}},$$

where  $x = a/b \in (0, 1)$ .

Necessity. If  $I(a, b) < Z_p(a, b)$  is true, then we have

$$\lim_{x \rightarrow 1} \frac{e^{-1}x^{x/(x-1)} - x^{\frac{x^p}{x^p+1}}}{(x-1)^2} = \frac{1}{12} - \frac{1}{4}p \leq 0,$$

which yields  $p \geq 1/3$ .

Sufficiency. It has been proved in [20, (5.7)] that  $I < Z_{1/3}$ . By the monotonicity proved in Theorem 2, it is derived that  $I < Z_{1/3} \leq Z_p$  if  $p \geq 1/3$ .

This completes the proof.  $\square$

Lastly, we will show that the inequality  $Z_{2/3} < Y$  is the best.

**Theorem 7.** For  $a, b > 0$  with  $a \neq b$ , the inequality  $Z_p < Y$  if and only if  $p \leq 2/3$ .

*Proof.* We assume that  $a < b$ . Then inequality  $Z_p(a, b) < Y(a, b)$  is equivalent with

$$(3.5) \quad x^{\frac{x^p}{x^p+1}} < e^{-1}x^{x/(x-1)} \exp\left(1 - \frac{x \ln^2 x}{(x-1)^2}\right),$$

where  $x = a/b \in (0, 1)$ .

Necessity. If  $Z_p(a, b) < Y(a, b)$  is valid, then we have

$$\lim_{x \rightarrow 1} \frac{\ln Z_p(x, 1) - \ln Y(x, 1)}{(x-1)^2} = \lim_{x \rightarrow 1} \frac{\frac{x^p}{x^p+1} \ln x - \frac{x-1}{x-1} \ln x + 1 - \left(1 - \frac{x \ln^2 x}{(x-1)^2}\right)}{(x-1)^2} = \frac{1}{4}p - \frac{1}{6} \leq 0,$$

which leads to  $p \leq 2/3$ .

Sufficiency. It has been proved in [20, (5.12)] that  $Z_{2/3} < Y$ . By the monotonicity of  $Z_p$  in  $p$ , it is derived that  $Y > Z_{2/3} \geq Z_p$  if  $p \leq 2/3$ .

This completes the proof.  $\square$

#### 4. REMARKS AND A CONJECTURE

**Remark 1.** With  $a^p \rightarrow a$ ,  $b^p \rightarrow b$ , the Lemma 4 can be restated as:

For  $a, b > 0$  with  $a \neq b$ , the inequalities  $P < N_p$  holds if and only if  $p \geq 1/2$ .

**Remark 2.** In the same way, the Lemma 5 is equivalent with:

For  $a, b > 0$  with  $a \neq b$ , the inequalities  $N_p < He$  holds if and only if  $p \leq 1/2$ .

**Remark 3.** From Theorem 3, Remark 1 and 2 the chain of inequalities for means (1.28) can be refined as

$$(4.1) \quad L_2 < P < N_{1/2} < He < A_{2/3} < I < Z_{1/3} < Y_{1/2}.$$

Also, it is clear that all the constants located in lower right corner are the best.

The chain of inequalities for means (4.1) is very nice. Unfortunately, it is not contain the second power-type Seiffert mean  $T_p$ . From (1.22) and (1.28) it is easy to obtained that

$$(4.2) \quad T_{2/5} < He < A_{2/3} < I < Z_{1/3} < Y_{1/2}.$$

If  $N_{1/2} < T_{2/5}$  holds, then we can get a more nice chain of inequalities for power-type means

$$(4.3) \quad L_2 < P < N_{1/2} < T_{2/5} < He < A_{2/3} < I < Z_{1/3} < Y_{1/2}.$$

Computation by mathematical software yields that if  $N < T_{4/5}$  then

$$\lim_{x \rightarrow 1} \frac{N(x, 1) - T_p(x, 1)}{(x - 1)^2} = \frac{1}{6} - \frac{5}{24}p \leq 0,$$

which implies that  $p \geq 4/5$ . And, we have

$$N(0^+, 1) - T_{4/5}(0^+, 1) = \frac{1}{2 \ln(\sqrt{2} + 1)} - \left(\frac{2}{\pi}\right)^{5/4} < 0.$$

Hence we propose the following conjecture:

**Conjecture 1.** For  $a, b > 0$  with  $a \neq b$ , the inequalities  $N < T_p$  holds if and only if  $p \geq 4/5$ .

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